

Topological dynamics of automorphism groups of countably categorical structures

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Abstract. We consider automorphism groups of some countably categorical structures and their precompact expansions. We prove that automorphism groups of ω -stable ω -categorical structures have metrizable universal minimal flows. We also study amenability of these groups.

2010 Mathematics Subject Classification: 03E15, 03C15

Keywords: Amenable groups, Countably categorical structures, G-flows.

0 Introduction

A group G is called **amenable** if every G -flow (i.e. a compact Hausdorff space along with a continuous G -action) supports an invariant Borel probability measure. If every G -flow has a fixed point then we say that G is **extremely amenable**. Let M be a relational structure which is a Fraïssé limit of a Fraïssé class \mathcal{K} . In particular \mathcal{K} coincides with $\text{Age}(M)$, the class of all finite substructures of M . By Theorem 4.8 of the paper of Kechris, Pestov and Todorcevic [16] the group $\text{Aut}(M)$ is *extremely amenable if and only if the class \mathcal{K} has the Ramsey property and consists of rigid*

*The author is supported by Polish National Science Centre grant DEC2011/01/B/ST1/01406

elements. Here the class \mathcal{K} is said to have the **Ramsey property** if for any k and a pair $A < B$ from \mathcal{K} there exists $C \in \mathcal{K}$ so that each k -coloring

$$\xi : \binom{C}{A} \rightarrow k$$

is monochromatic on some $\binom{B'}{A'}$ from C which is a copy of $\binom{B}{A}$, i.e.

$$C \rightarrow (B)_k^A.$$

In the situation when \mathcal{K} does not have Ramsey property one can consider **Ramsey degrees** of A 's defined as the minimal k such that for every $r \in \omega$ and $B \in \mathcal{K}$ with non-empty $\binom{B}{A}$ there exists $C \in \mathcal{K}$ so that each r -coloring

$$\xi : \binom{C}{A} \rightarrow r$$

is $(\leq k)$ -chromatic on some $\binom{B'}{A'}$ from C which is a copy of $\binom{B}{A}$.

We remind the reader that a G -flow X is called **minimal**, if every its G -orbit is dense. The flow X is **universal**, if for every G -flow Y there is a continuous G -map $f : X \rightarrow Y$. According to topological dynamics a universal minimal flow always exists and is unique up to G -flow isomorphism (and is usually denoted by $M(G)$). The following question was formulated by several people. In particular it appears in the paper of Angel, Kechris and Lyons [2].

Let $G = \text{Aut}(M)$, where M is a countably categorical structure. Is the universal minimal G -flow metrizable?

Recently A.Zucker has found a characterisation of automorphism groups of relational structures which have metrizable universal minimal flows. It substantially develops the previous work of Kechris, Pestov, Todorcevic and Nguyen van Thé from [16] and [19].

Theorem A (Theorem 1.2 of [25]). *Let M be a relational structure which is a Fraïssé limit of a Fraïssé class \mathcal{K} . Then the following are equivalent.*

- 1) $G = \text{Aut}(M)$ has metrizable universal minimal flow,
- 2) each $A \in \mathcal{K}$ has finite Ramsey degree,
- 3) there is a sequence of new relational symbols \bar{S} and a precompact \bar{S} -expansion of M , say M^* , so that

- (i) M^* is a Fraïssé structure,
- (ii) $\text{Aut}(M^*)$ is extremely amenable and
- (iii) the closure of the G -orbit of M^* in the space of \bar{S} -expansions of M is a universal minimal G -flow.

Moreover if $M(G)$ is metrizable, then G has the **generic point property**, i.e. $M(G)$ has a G_δ -orbit.

In this formulation precompactness means that every member of \mathcal{K} has finitely many expansions in $\text{Age}(M^*)$.

By this theorem it is crucial to know whether there is a countably categorical structure M which does not have expansions as in Theorem A. It is worth noting that some versions of this question were formulated for example in [4], see Problems 27, 28. Related results can be also found in [17], [2] and [24].

We also mention the following related questions from [2].

1. Describe Polish groups G so that the universal minimal G -flow is metrizable.
2. **Conjecture.** Let G be Polish and $M(G)$ be metrizable. Then $M(G)$ has a G_δ -orbit (i.e. the generic point property holds).

These questions are also open for amenable G .

In our paper having in mind these respects, we consider automorphism groups of countably categorical structures which satisfy some standard model-theoretic properties, see [20]. We will prove in Section 2.1 that the automorphism group of an ω -stable ω -categorical structure has metrizable universal minimal flow and thus by Theorem A this group satisfies the generic point property. In some typical cases such groups are amenable (see Section 2.2).

We also discuss possible extensions of these results to smoothly approximable structures (Section 3.1) and structures defined on the Urysohn space (Section 3.2). In particular we describe a very flexible construction which associates to any Fraïssé structure M which is ω -categorical, a structure defined on \mathbb{U} by some continuous predicates. In cases when the universal minimal Aut-flow of the obtained extension \mathbb{U}_M exists it coincides with the corresponding flow for $\text{Aut}(M)$.

We slightly modify the approach from [16], [19] and [25] to extreme amenability so that it works for structures where elimination of quantifiers is not necessarily satisfied, for example obtained by Hrushovski's amalgamation method. This brings additional flexibility. Here we use [14] and [18], see Section 1.

1 Truss' condition and the Ramsey property

Let \mathcal{K} be a universal class of finite structures of some countable language L . We assume that \mathcal{K} is the age of some countable uniformly locally finite structure. In particular \mathcal{K} satisfies JEP.

Let \mathbf{X} be the space of all L -structures M on the set ω so that the age of M is contained in \mathcal{K} . It is a closed subset of the complete metric space of all L -structures on ω under the standard topology [18]. Thus \mathbf{X} is complete and the Baire Category Theorem holds for \mathbf{X} .

It is also clear that S_∞ acts continuously on \mathbf{X} with respect to our topology. We say that $M \in \mathbf{X}$ is *generic* if the class of its images under S_∞ is comeagre in \mathbf{X} .

The following definition was introduced in [14] and [18] in a much more general situation of expansions of countably categorical structures.

The class \mathcal{K} has the *weak amalgamation property* (see [18], in the original paper [14] it is called the *almost amalgamation property*) if for every $A \in \mathcal{K}$ there is an extension $A' \in \mathcal{K}$ such that for any $B_1, B_2 \in \mathcal{K}$, extending A' , there exists a common extension $D \in \mathcal{K}$ which amalgamates the corresponding maps $A \rightarrow B_i$, $i = 1, 2$.

Theorem B. ([14], Theorem 1.2 and Corollary 1.4) *The set \mathbf{X} has a generic structure if and only if \mathcal{K} has the weak amalgamation property.*¹

It is worth noting that the age of the generic structure coincides with \mathcal{K} . Let us fix such a structure M . We will usually assume that M is ω -categorical.

Remark 1.1 By the proof of Theorem 1.2 ($1 \rightarrow 2$) of [13] the weak amalgamation property is a consequence of the following version of the Ramsey property:

For any $A \in \mathcal{K}$ there is an extension $A' \in \mathcal{K}$ such that for any $B_1 \in \mathcal{K}$, where $A' \leq B_1$, there exists an extension $B_1 < B_2 \in \mathcal{K}$ such that

$$B_2 \rightarrow (B_1)_2^A.$$

An element $A \in \mathcal{K}$ is called an *amalgamation base* if any two of its extensions have a common extension in \mathcal{K} under some embeddings fixing A . We say that \mathcal{K} satisfies *Truss' condition* if any element of \mathcal{K} extends to an amalgamation base. If it holds then the set of amalgamation bases is a cofinal subset of \mathcal{K} which has the amalgamation property. It is easy to see that Truss' condition is equivalent to existence of a cofinal subfamily $\mathcal{C} \subset \mathcal{K}$ which satisfies JEP and AP. It is also clear that Truss' condition implies the weak amalgamation property. In particular it implies the existence of a generic structure. In this case we also have the following characterisation (for example see [7]):

A countable structure M with $\text{Age}(M) = \mathcal{K}$ is generic if and only if for any pair $A < B$ from \mathcal{C} any embedding of A into M extends to an embedding of B into M .

It is worth noting that in this case any partial isomorphism of M between two substructures from \mathcal{C} extends to an automorphism of M . Assuming that for every n the class \mathcal{K} has finitely many n -generated substructures we obtain that $\text{Th}(M)$ is ω -categorical and model complete.

The following theorem is a slightly generalized version of Theorem 4.5 from [16].

Theorem 1.2 *Let \mathcal{K} satisfy Truss' condition. Let $\mathcal{C} \subset \mathcal{K}$ be a cofinal subset of amalgamation bases with the joint embedding property and the amalgamation property.*

Then the automorphism group $\text{Aut}(M)$ of a generic structure is extremely amenable if and only if the class \mathcal{C} has the Ramsey property and consists of rigid elements.

¹We should mention that a related property, so called *density of maximal \exists -types*, was considered by W.Hodges in [11].

In fact this theorem coincides with Theorem 5.1 of [25]. We give a small comment concerning this. A. Zucker in [25] considers the main properties of the KPT-theory in terms of embeddings. In particular the Ramsey property for embeddings is formulated as follows.

Definition 1.3 *The class \mathcal{K} is said to have the **Ramsey property for embeddings** if for any k and a pair $A < B$ from \mathcal{K} there exists $C \in \mathcal{K}$ so that each k -coloring of embeddings of A into C*

$$\xi : \text{Emb}(A, C) \rightarrow k$$

is monochromatic on some $\text{Emb}(A, B')$ where B' is a copy of B in C . It is denoted by

$$C \hookrightarrow (B)_k^A.$$

Now it is clear that the condition that the class \mathcal{C} has the Ramsey property for embeddings (as in Theorem 5.1 of [25]) is a reformulation of the statement " \mathcal{C} has the Ramsey property and consists of rigid elements" in Theorem 1.2.

It is also clear how to define the **embedding Ramsey degree** of a structure A in \mathcal{K} (also see Section 4 of [25]). By Proposition 4.4 of [25] A has finite Ramsey degree in \mathcal{K} if and only if A has finite embedding Ramsey degree in \mathcal{K} . In particular condition 2) of Theorem A is equivalent to the condition that each $A \in \mathcal{K}$ has finite embedding Ramsey degree.

Let us consider the situation of Theorem 1.2 again. By Proposition 4.6 of [25] each $A \in \mathcal{C}$ has the same embedding Ramsey degree both in \mathcal{C} and in \mathcal{K} . It is worth noting that the following general statement holds.

Lemma 1.4 *if \mathcal{C} is a cofinal subset of \mathcal{K} , then any $A \in \mathcal{K}$ has finite Ramsey degree in \mathcal{K} if and only if any $B \in \mathcal{C}$ has finite Ramsey degree in \mathcal{C} .*

Proof. We only need to prove that in the situation $A < B$ with $B \in \mathcal{C}$ the embedding Ramsey degree of A in \mathcal{K} is not greater than the embedding Ramsey degree of B in \mathcal{C} multiplied by the number of embeddings of A into B . This is easy. \square

2 ω -Stable ω -categorical structures

2.1 Metrizability of universal minimal flows

In this section we prove the following theorem.

Theorem 2.1 *Let M be an ω -stable countably categorical structure. Then M has a precompact expansion M' so that $\text{Aut}(M')$ is extremely amenable and the closure of $\text{Aut}(M) \cdot M'$ is the universal minimal $\text{Aut}(M)$ -flow. In particular $\text{Aut}(M)$ has the generic point property.*

We need some preliminary material from Sections 2 and 3 of [20].

By Section 3.2 of [20] any transitive ω -stable ω -categorical structure N can be presented (up to bi-interpretability) in the form of "a tree structure" as follows. The structure N consists of n pairwise disjoint levels $L_1 \cup \dots \cup L_n$ with a sequence of projections $\pi_i : L_{i+1} \rightarrow L_i$, $i \leq n-1$, so that

- for each $i \leq n-1$ and $a \in L_{i+1}$ the type $tp(a/\pi_i(a))$ is algebraic or strictly minimal,
- if $tp(a/\pi_i(a))$ is strictly minimal and affine then it is not orthogonal to some $tp(\pi_{ij}(a)/\pi_{i(j-1)}(a))$ for $j < i$, where π_{ij} maps L_{i+1} to L_j by iterations of appropriate π_l ,
- if $tp(a/\pi_i(a))$ is strictly minimal and projective then it is stationary.

We thus may assume that the structure M from the formulation of the theorem is given in this form as a relational structure with all structure induced by M^{eq} . It is worth noting here that any ω -categorical structure is bi-interpretable with a theory with a unique 1-type (Lemma 3.8 of [12]). By [1] these structures have the same automorphism groups considered as topological groups.

We assume that M consists of finitely many sorts (it is called **regularity**), admits elimination of quantifiers and contains a copy of each canonical projective geometry which is non-orthogonal to a coordinatizing geometry of M (i.e. the language is **adequate**). The set $\{1, 2, \dots, n\}$ is divided into four parts as follows:

- I_{new} consists of $i < n$ where $tp(a/\pi_i(a))$ is projective or trivial and orthogonal to all $tp(a'/\pi_j(a'))$ with $j < i$,
- I_{old} consists of $i < n$ where $tp(a/\pi_i(a))$ is projective or trivial and non-orthogonal to some $tp(a'/\pi_j(a'))$ with $j < i$,
- I_{aff} consists of $i < n$ where $tp(a/\pi_i(a))$ is affine,
- I_{fin} consists of $i < n$ where $tp(a/\pi_i(a))$ is algebraic.

For $i \in I_{old}$ there is a 0-definable relation defining a function $f_i(x, y)$ witnessing non-orthogonality of $tp(a/\pi_i(a))$ with $tp(\pi_{ij}(a)/\pi_{i(j-1)}(a))$ where $j < i$ and is minimal. For $b \in L_i$ the function $f_i(b, -)$ bijectively maps the set of realisations of $tp(\pi_{ij}(b)/\pi_{i(j-1)}(b))$ which are outside of $acl(b)$ to the set of realisations of $tp(a/b)$ with $\pi_i(a) = b$.

Following Construction 2.4 of Section 3.2 of [20] one can also build for each $i \in I_{aff}$ a 0-definable relation defining a function $f_i(x, \bar{z}, -, -, -)$ witnessing the non-orthogonality mentioned above. Here x corresponds to elements of L_i and \bar{z} corresponds to tuples of affine lines (consisting of z_k with $\pi_i(z_k) = x$) and $f_i(x, \bar{z}, -, -, -)$ maps appropriate triples of L_j as above to L_{i+1} .

If the theory is unidimensional (i.e. totally categorical) then it has the following structure. By Lemma 2.6.10 of [20] we may assume L_1 is a modular strictly minimal set. Let us denote it by D . The assumption of total categoricity gives that all non-algebraic types appearing in the construction are not orthogonal to D .

Repeating Definition 2.6.11 of [20] we call $E \subset M$ a **D -envelope**, if for some $A \subset M$ the set E is maximal with respect to the conditions $A \subseteq E$ and $\text{acl}(E) \cap D = \text{acl}(A) \cap D$. By Section 2.6 of [20]

- D -envelopes are homogeneous, i.e. tuples of the same type in M^{eq} are in the same orbit of envelope's automorphisms,
- D -envelopes of finite subsets are finite and
- each finite subset of M is contained in a finite D -envelope.

If the theory is not unidimensional, then envelopes are introduced according to Section 3.1 of [6]. We give a brief description of it (which is not complete). Structure M is considered in a regular adequate eq -expansion. Let μ be a **dimension function** of $Th(M)$, i.e. μ associates to each equivalence class of standard systems of projective geometries a number from ω , a finite dimension of this type of geometries. Then μ -**envelope** is a subset E satisfying the following three conditions:

- (i) E is algebraically closed in M ,
- (ii) for $c \in M \setminus E$ there is a standard system of geometries J with domain A and an element $b \in A \cap E$ for which $\text{acl}(Ec) \cap J_b$ properly contains $\text{acl}(E) \cap J_b$,
- (iii) for J a standard system of geometries defined on A and $b \in A \cap E$, $J_b \cap E$ has the isomorphism type given by $\mu(J)$.

As in the totally categorical case μ -envelopes are finite, unique and homogeneous. The latter means that any elementary map between two subsets of E extends to an automorphism of E which is elementary in M . Moreover envelopes are cofinal in the set of finite substructures of M (for appropriate μ).

Proof of Theorem 2.1. We preserve the notation above. Consider the totally categorical case. We distinguish this case because it will be presented in a complete form. Since the general case is treated in a similar way we will only briefly describe it.

We know that the family \mathcal{C} of all finite D -envelopes is cofinal in the class \mathcal{K} of all finite substructures of M and has the joint embedding property. The amalgamation property can be shown as follows. If $f_1 : A \rightarrow B_1$ and $f_2 : A \rightarrow B_2$ are embeddings of finite D -envelopes, then taking a D -envelope C extending B_1 and B_2 , we satisfy the amalgamation property by applying homogeneity of C in order to find appropriate embeddings of B_i into C . By Theorem B we see that there is a \mathcal{K} -generic structure where \mathcal{C} is the appropriate family of amalgamation bases. By the properties of M collected above it is clear that M is the corresponding generic.

Claim 1. The class \mathcal{C} has the Ramsey property.

Indeed any embedding between D -envelopes is obtained by lifting of the corresponding maps of their D -parts. Moreover these D -parts uniquely determine their envelopes. Thus the Ramsey property for \mathcal{C} is equivalent to the Ramsey property for the family of finite algebraically closed subsets of D . Since D is a pure set or a projective geometry over a finite field, the corresponding Ramsey property follows from well-known theorems of Ramsey theory, for example see [21].

We conclude this case by applying condition 2) of Theorem 1.2 of [25] (Theorem A above).

Let us consider the case of ω -stable ω -categorical structures in general. Let \mathcal{E}_{const} be the family of all finite μ -envelopes where μ is a constant function: μ has the same value for any type of a geometry. It is clear that \mathcal{E}_{const} is cofinal in the class \mathcal{K} of all finite substructures of M and has the joint embedding property. The amalgamation property can be shown as follows. Let $f_1 : A \rightarrow B_1$ and $f_2 : A \rightarrow B_2$ be embeddings of finite envelopes, with constant dimension functions μ_0, μ_1 and μ_2 respectively. Let $\mu = \mu_1 + \mu_2$. Take a μ -envelope C extending B_1 and B_2 . Then the amalgamation property is verified by applying homogeneity of C in order to find appropriate embeddings of B_i into C . By Theorem B we see that there is a \mathcal{K} -generic structure for \mathcal{E}_{const} as the appropriate family of amalgamation bases. By the properties above it is clear that M is the corresponding generic.

Claim 2. The class \mathcal{E}_{const} has the Ramsey property.

Indeed any embedding of a μ -envelope into a μ' -envelope from \mathcal{E}_{const} (where $\mu < \mu'$) is uniquely defined by lifting of the corresponding maps between geometries determined by μ and μ' . Thus the Ramsey property for \mathcal{E}_{const} is equivalent to the Ramsey property for the family of finite algebraically closed subsets of geometries involved into M . Since such a geometry is a pure set or a projective geometry over a finite field, the corresponding Ramsey property follows from Ramsey theory, for example see [21]. \square

2.2 Amenability of the automorphism group

The theorem of Kechris, Pestov and Todorcevic mentioned in Introduction has become a basic tool to amenability of automorphism groups. Even before Theorem A appeared, a standard approach to verifying whether $Aut(M)$ is amenable was based on looking for an expansion M^* of M exactly as in Theorem A, see [16], [17], [19], [2] and [24] (were even some weak versions of Theorem A occur). Theorem 9.2 from [2] and Theorem 2.1 from [24] describe amenability of $Aut(M)$ in this situation.

Thus the results of Section 2.1 naturally lead us to the following conjecture.

Conjecture. Let M be an ω -stable countably categorical structure. Then $Aut(M)$ is amenable.

By Theorem 3.1 of [12] M is a reduct of an ω -stable countably categorical structure M' such that the theory $Th(M')$ is nonmultidimensional. By [1] this means that there is a continuous homomorphism from $Aut(M')$ into $Aut(M)$. Thus it is natural to start with the nonmultidimensional case. Let us assume a stronger property that M is **unidimensional**, i.e. $Th(M)$ is totally categorical. The following definitions and statements give some basic information about this case.

Let M be an ω -stable ω -categorical structure. If P and Q are 0-definable sets in M^{eq} we define Q is a **precover** of P if there are

- (a) a partition of $Q \setminus P$ into a 0-definable family $\{H_{\bar{a}} : \bar{a} \in P\}$,
- (b) a 0-definable family $\{\Gamma_{\bar{a}} : \bar{a} \in P\}$ of groups (the **structure groups**)

in P^{eq} ,

(c) a regular \bar{a} -definable action of each $\Gamma_{\bar{a}}$ on $H_{\bar{a}}$.

We now state Zilber's "ladder theorem".

Theorem C. ([23], but we follow [9], p.14) *Let M be totally categorical. Then there is a 0-definable modular strictly minimal set D and a sequence*

$$D = M_0 \subset M_1 \subset \dots \subset M_n$$

such that each M_{i+1} is a precover of M_i and M is in the definable closure of M_n . Furthermore all structure groups live in D^{eq} and they are finite or vector spaces over \mathbf{F}_q , where the latter case occurs only when D is a projective space over \mathbf{F}_q .

Let us consider the case when M is in the algebraic closure of D .

Proposition 2.2 *Let M be a countable totally categorical structure which lies in the algebraic closure of some 0-definable modular strictly minimal set D in M^{eq} .*

Then $\text{Aut}(M)$ is an amenable group.

Proof. Assume that M is a structure of Morley rank n . By Theorem 3.2 of [8] there exists a finite 0-definable subset M_0 with $\text{acl}^{eq}(\emptyset) = \text{dcl}^{eq}(M_0)$, and a sequence

$$M_0 \cup D \subseteq M_{1,0} \subseteq M_1 \subseteq M_{2,0} \subseteq \dots \subseteq M_{n,0} \subseteq M_n \supseteq M$$

such that

- (i) M_i has Morley rank i ,
- (ii) $\text{Aut}(M_{1,0}/M_0 \cup D)$ is nilpotent-by-finite-abelian,
- (iii) for $2 \leq i \leq n$ $\text{Aut}(M_{i,0}/M_{i-1})$ is nilpotent, and for $1 \leq i \leq n$ $\text{Aut}(M_i/M_{i,0})$ is a direct product of finite groups.

Since S_∞ and the automorphism group of an ω -dimensional vector space over a finite field are amenable ([2]), the group of automorphisms of $M_0 \cup D$ induced by $\text{Aut}(M)$ is amenable too. It remains to prove that $\text{Aut}(M/D \cup M_0)$ is amenable. The latter is reduced to proving of amenability of groups $\text{Aut}(M_{1,0}/M_0 \cup D)$, $\text{Aut}(M_{i,0}/M_{i-1})$ for $2 \leq i \leq n$, and $\text{Aut}(M_i/M_{i,0})$ for $1 \leq i \leq n$. Since all of them are soluble or compact, the rest is clear. \square

The following theorem slightly generalises Proposition 2.2.

Theorem 2.3 *Let M be an ω -stable ω -categorical structure having an expansion to a totally categorical structure which lies in the algebraic closure of some 0-definable modular strictly minimal set D .*

Then $\text{Aut}(M)$ is an amenable group.

Proof. The argument of the proof of Theorem 4.10 from [12], p. 157, together with the proof of Proposition 2.2 show that $Aut(M)$ has a topological Jordan-Hölder sequence

$$\{1\} = G_0 \subseteq G_1 \subseteq \dots \subseteq G_n = Aut(M),$$

such that for each i the group G_{i+1}/G_i is isomorphic as a topological group to one of the following:

- (i) a finite group,
- (ii) a soluble group,
- (iii) S_∞ or $PGL(\omega, \mathbf{F}_q)$ for some fixed q ,
- (iv) the product H^ω where H is as in (i), (ii), (iii) respectively.

Since all these groups are amenable, $Aut(M)$ is amenable too. \square

3 Possible extensions

In Section 3.1 we consider the question if the results of Section 2 can be extended to smoothly approximate structures. In Section 3.2 we consider a similar question in the case of some structures defined on the Urysohn space.

3.1 Ramsey property, independence and amalgamation

Let M be the Fraïssé limit of a Fraïssé class \mathcal{K} . Let \mathcal{P} be a family of types over \emptyset so that for every $n \in \omega \setminus \{0\}$ the family \mathcal{P} contains n -types and if $t(x_1, \dots, x_n) \in \mathcal{P}$ then for any permutation $\sigma \in S_n$ the type $t(\sigma(\bar{x}))$ belongs to \mathcal{P} . We do not assume that types are complete.

Definition 3.1 *We call \mathcal{P} a **freeness relation** if the following property holds.*

Let a_1, a_2, \dots, a_n and b_1, b_2, \dots, b_k be sequences from M which realise types from \mathcal{P} . Then there is a sequence $a_1, a_2, \dots, a_n, a'_1, a'_2, \dots, a'_k \in M$ realising a type from \mathcal{P} , where tuples a'_1, a'_2, \dots, a'_k and b_1, b_2, \dots, b_k are of the same quantifier free type.

As an example of this situation consider infinite dimensional vector spaces V over a finite field F . Then types of independent sequences form a freeness relation. Some other examples of this freeness relation can be obtained by adding appropriate bilinear forms.

In general we may assume that M is given with a notion of independence of two subsets over a third so that some standard axioms of forking independence are satisfied, see [20]. In fact we need invariance with respect to elementary maps, symmetry existence and extension (transitivity is not necessary). Then types of independent sequences over \emptyset form a freeness relation.

Definition 3.2 We say that the freeness relation of M satisfies **JN-amalgamation** if for every free sequence of elements a_1, a_2, \dots, a_k there is a finite family \mathcal{F} of tuples \bar{c} of type \bar{a} so that the following conditions are satisfied:

- any two distinct tuples from \mathcal{F} do not have a common pair of elements;
- for every linear ordering $<$ of $\bigcup \mathcal{F}$ there exists $\bar{c} \in \mathcal{F}$ so that $<$ defines the enumeration of \bar{c} .

The paper of J.Jezek and J.Nesetril [15] contains natural example of structures where JN-amalgamation holds. For example Lemma 3.5 of that paper says that a pure infinite set has this property.

We now introduce some technical property.

Definition 3.3 We say that a free sequence of elements a_1, a_2, \dots, a_k is **strict** in M if any finite substructure $C < M$ has an order $<$ so that for any two tuples \bar{c}_1 and \bar{c}_2 of type \bar{a} which generate the same substructure of C the map from \bar{c}_1 to \bar{c}_2 preserving $<$ is elementary.

It is clear that this property holds if the subset $\{a_1, a_2, \dots, a_k\}$ is uniquely determined by a type of (any) its enumeration in the substructure generated by it. Then any linear order works.

Theorem 3.4 Let M be the Fraïssé limit of a Fraïssé class \mathcal{K} . We assume that M is given with a freeness relation having JN-amalgamation. If the class \mathcal{K} satisfies the Ramsey property then the type of any strict free sequence \bar{a} from M is the same for all permutations of \bar{a} .

Proof. The proof is based on the argument of Proposition 3.6 from [15]. Suppose that \bar{a} is strict and a permutation p of \bar{a} does not preserve the type of \bar{a} . By the definition of freeness relations there is a free sequence $\bar{a}\bar{a}'$, where \bar{a}' is a copy of \bar{a} . We define a linear ordering \prec of $\bar{a}\bar{a}'$ as follows. The tuple \bar{a} is an initial segment where \prec is defined by the enumeration of \bar{a} . In the final segment \bar{a}' we put $a'_i \prec a'_j$ if $p(i) < p(j)$.

By JN-amalgamation there is a finite family \mathcal{F} of tuples \bar{c} of type $\bar{a}\bar{a}'$ so that the following conditions are satisfied:

- any two distinct tuples from \mathcal{F} do not have a common pair of elements;
- for every linear ordering $<$ of $\bigcup \mathcal{F}$ there exists $\bar{c} \in \mathcal{F}$ so that $<$ defines a copy of \prec on \bar{c} .

Let B be a finite substructure of M containing \mathcal{F} and let A be the structure generated by \bar{a} . To show that \mathcal{K} does not have the Ramsey property take any $C < M$ with $B < C$ and fix any linear ordering $<$ of C which witnesses strictness of \bar{a} .

We color $A' \in \binom{C}{A}$ white if for any copy of \bar{a} , say \bar{b} , generating A' the type of \bar{b} with respect to $<$ coincides with the type of \bar{a} . In the contrary case we color A' black.

Now note that for any $B' \in \binom{C}{B}$ we find some $\bar{c} \in B'$ of type $\bar{a}\bar{a}'$ so that $<$ induces a copy of \prec on \bar{c} . Thus the substructure generated by the initial segment of \bar{c} has a different color compared with the substructure generated by the final part of \bar{c} . \square

Note that in the case of vector spaces with bilinear forms defining classical geometries (symplectic, unitary or orthogonal) permutations of tuples usually do not preserve the type. We do not know if these spaces have any property similar to JN-amalgamation. If this is the case we conjecture that the results of Section 2.1 cannot be extended to smoothly approximable structures. We think that arguments of the theorem above would refute condition (2) of Theorem A.

3.2 Expansions of the Urysohn space

Let \mathbb{U} be the Urysohn space of diameter 1. This is the unique Polish metric space which is universal and ultrahomogeneous, i.e. every isometry between finite subsets of \mathbb{U} extends to an isometry of \mathbb{U} . The space \mathbb{U} is considered in the continuous signature $\langle d \rangle$. It is known that $\text{Iso}(\mathbb{U})$ is extremely amenable [16].

The countable counterpart of \mathbb{U} is the *rational Urysohn space of diameter 1*, \mathbb{QU} , which is both ultrahomogeneous and universal for countable metric spaces with rational distances and diameter ≤ 1 . It is shown in Section 5.2 of [3] that there is an embedding of \mathbb{QU} into \mathbb{U} so that:

- (i) \mathbb{QU} is dense in \mathbb{U} ;
- (ii) any isometry of \mathbb{QU} extends to an isometry of \mathbb{U} and $\text{Iso}(\mathbb{QU})$ is dense in $\text{Iso}(\mathbb{U})$;
- (iii) for any $\varepsilon > 0$, any partial isometry h of \mathbb{QU} with domain $\{a_1, \dots, a_n\}$ and any isometry g of \mathbb{U} such that $d(g(a_i), h(a_i)) < \varepsilon$ for all i , there is an isometry \hat{h} of \mathbb{QU} that extends h and is such that for all $x \in \mathbb{U}$, $d(\hat{h}(x), g(x)) < \varepsilon$.

The space \mathbb{QU} is usually considered as the first-order structure of infinitely many binary relations

$$d(x, y) \leq q, \text{ where } q \in \mathbb{Q} \cap [0, 1].$$

This language will be denoted by L_0 .

Let now L be an arbitrary countable first-order language and \mathcal{K}_0 be a universal class of finite L -structures which satisfies Truss' condition. Let \mathcal{C}_0 be a cofinal subfamily with the joint embedding property and the amalgamation property. Let M be the generic L -structure with respect to \mathcal{C}_0 , i.e. $\text{Age}(M) = \mathcal{K}_0$ and M is \mathcal{C}_0 -homogeneous: any isomorphism in M between finite substructures from \mathcal{C}_0 extends to an automorphism of M .

Let \mathcal{K}_M be the (universal) class of all finite structures F of the language $L_0 \cup L \cup \{P^M\}$, where:

- F is an L_0 -metric space of diameter ≤ 1 ;
- any two distinct elements of P^M are at the distance 1;
- the predicate P^M defines an L -substructure from \mathcal{K}_0 .

We assume that \mathcal{K}_M contains the class \mathcal{K} of all finite L_0 -metric spaces of diameter ≤ 1 considered as structures F with $P^M(F) = \emptyset$. On the other hand the L_0 -reducts of all structures from \mathcal{K}_M form \mathcal{K} too.

Lemma 3.5 *The subclass $\mathcal{C}_M \subseteq \mathcal{K}_M$ consisting of structures where P^M defines substructures of \mathcal{C}_0 is a cofinal subclass with the joint embedding property and the amalgamation property.*

Proof. Note that for any $F \in \mathcal{K}_M$ and any $A \in \mathcal{K}_0$ (considered as $\{0, 1\}$ -metric space) there is a natural free amalgamation of A and F over the common part $A \cap P^M(F)$ so that all elements of $A \setminus P^M(F)$ are at the distance 1 from F and satisfy P^M . This implies cofinality of \mathcal{C}_M .

We now demonstrate an argument for the JEP and AP. Assume that $F_1, F_2 \in \mathcal{C}_M$ and let $D \in \mathcal{C}_0$ gives AP (resp. JEP) of $P^M(F_1)$ and $P^M(F_2)$. Then we amalgamate D with F_1 and F_2 respectively. We obtain two structures \hat{F}_1 and \hat{F}_2 with $P^M(\hat{F}_1) \cap P^M(\hat{F}_2) = D$ and $\hat{F}_1 \cap \hat{F}_2 = (F_1 \cap F_2) \cup D$. Now amalgamating metrics as in Theorem 2.1 of [5] (and truncating it if necessary) we obtain the result. \square

By Theorem B of Section 1 the class \mathcal{K}_M has a generic structure. We call it \mathbb{QU}_M . Since \mathbb{QU}_M is \mathcal{C}_M -homogeneous, the P^M -part of this structure is generic with respect to \mathcal{C}_0 . In particular $P^M(\mathcal{C}_M)$ is isomorphic to M .

Lemma 3.6 *The metric spaces \mathbb{QU} and \mathbb{QU}_M are isometric.*

Proof. It is clear that any finite metric space over \mathbb{Q} is embeddable into \mathbb{QU}_M . Since \mathbb{QU}_M is a Fraïssé limit of \mathcal{K}_M it suffices to show that \mathbb{QU}_M is *finitely injective*, i.e. given finite metric spaces $A \subset \mathbb{QU}_M$ and $A \subset A'$ there is an isometric embedding of A' into \mathbb{QU}_M over A . This in turn is equivalent to the property that if $A \subset A'' \in \mathcal{K}_M$ then A'' extends to some $A''' \in \mathcal{K}_M$ with A' embeddable into A''' over A . The latter can be easily realised. Indeed by Theorem 2.1 of [5] freely amalgamating A'' with A' over A (i.e. with $(A' \setminus A) \cap P^M = \emptyset$) we obtain a required A''' . Indeed P^M defines in this amalgamation an L -structure from \mathcal{K}_0 . \square

From now on we will assume that M is a Fraïssé structure with respect to \mathcal{K}_0 , i.e. \mathcal{K}_0 can be taken as \mathcal{C}_0 .

Since the group $\text{Aut}(M)$ has metrizable universal minimal flow, the structure M has an expansion M^* which satisfies condition (3) of Theorem A. Let $T^* = Th(M^*)$ and let \mathcal{K}_0^* be the age of M^* .

By Lemma 3.5 applied to the class \mathcal{K}_{M^*} we obtain \mathbb{QU}_{M^*} where $P^M(\mathbb{QU}_{M^*})$ is isomorphic to M^* . We have $\text{Iso}(\mathbb{QU}_{M^*}) < \text{Iso}(\mathbb{QU}_M)$.

We need the following reformulation of condition (3) of Theorem A.

Theorem A' (Theorem 8.14 of [25]). *Let M be a relational structure which is a Fraïssé limit of a Fraïssé class \mathcal{K} . Then the following are equivalent.*

- 1) $G = \text{Aut}(M)$ has metrizable universal minimal flow,
- 2) each $A \in \mathcal{K}$ has finite Ramsey degree,
- 3) there is a sequence of new relational symbols \bar{S} and a precompact \bar{S} -expansion of M , say M^* , so that

- (i) M^* is a Fraïssé structure where $\text{Aut}(M^*)$ is extremely amenable,
- (ii) the class $\mathcal{K}^* = \text{Age}(M^*)$ is a **reasonable expansion** of \mathcal{K} , i.e. for any $A, B \in \mathcal{K}$, an embedding $f : A \rightarrow B$ and an expansion $A^* \in \mathcal{K}^*$ of A the embedding f also embeds A^* into some expansion $B^* \in \mathcal{K}^*$ of B and

(iii) the class \mathcal{K}^* has the **expansion property** with respect to \mathcal{K} , i.e. for every $A^* \in \mathcal{K}^*$ there is $B \in \mathcal{K}$ such that for any expansion $B^* \in \mathcal{K}^*$ of B the structure A^* embeds into B^* .

We assume that the pair of M and its expansion M^* chosen before the formulation of Theorem A' satisfies condition (3) of Theorem A'. Note that condition 3(ii) for $(\mathcal{K}_0^*, \mathcal{K}_0)$ immediately implies 3(ii) for the pair $(\mathcal{K}_{M^*}, \mathcal{K}_M)$.

Condition 3(iii) is also easy.

Lemma 3.7 *The class \mathcal{K}_{M^*} has the expansion property with respect to \mathcal{K}_M .*

Proof. Having $A^* \in \mathcal{K}_{M^*}$ choose $D \in \mathcal{K}_0$ so that for any expansion $D^* \in \mathcal{K}_0^*$ of D the structure $P^M(A^*)$ embeds into D^* . Let $A \in \mathcal{K}_M$ be the reduct of A^* . Let f_1, \dots, f_t be a sequence of all embeddings of $P^M(A)$ into D . At the first step we amalgamate A with D with respect to the embedding f_1 exactly as in Lemma 3.5. Let B_1 be the obtained structure. Now amalgamate B_1 with A where $P^M(A)$ meets B_1 by the image of f_2 . Continuing this procedure we obtain B_t after t steps. Note that $P^M(B_t) = D$. Now it is easy to see that this structure satisfies the statement of the lemma. \square

The following question is non-trivial.

Is $\text{Iso}(\text{QU}_{M^*})$ extremely amenable?

In fact this is exactly the question if the pair $(\text{QU}_{M^*}, \text{QU}_M)$ (and $(\mathcal{K}_{M^*}, \mathcal{K}_M)$) satisfies condition (3) of Theorem A'. We conjecture that even in the situation of ω -stable ω -categorical M this happens rather rarely. On the other hand we do not have any example where it does not hold.

In cases when the answer is positive the group $\text{Iso}(\text{QU}_M)$ has metrizable universal minimal flow. By Theorem 5.7 of [25] it is realised by the space of all expansions of QU_M which have ages $\subseteq \text{Age}(\text{QU}_{M^*})$.

On the other hand note that the group $\text{Iso}(\text{QU}_M)$ also has a natural actions on the space $\text{Exp}(M, \text{Age}(M^*))$ of all expansions of M which have ages $\subseteq \text{Age}(M^*)$. These action is defined by restriction to P^M .

Lemma 3.8 *If $\text{Iso}(\text{QU}_M)$ has metrizable universal minimal flow, then it is isomorphic to the space $\text{Exp}(M, \text{Age}(M^*))$.*

Proof. By the definition of the class \mathcal{K}_M it is easy to see that any map between isomorphic substructures of $P^M(\text{QU}_M)$ (which is M) extends to an isometry of QU_M . This means that the closure of the orbit $\text{Iso}(\text{QU}_M) \cdot M^*$ coincides with $\text{Exp}(M, \text{Age}(M^*))$.

On the other hand note that any element of $\text{Exp}(M, \text{Age}(M^*))$ can be naturally identified with its extension to an element of $\text{Exp}(\text{QU}_M, \text{Age}(\text{QU}_{M^*}))$. In this way the orbit $\text{Iso}(\text{QU}_M) \cdot M^*$ is identified with elements of the orbit $\text{Iso}(\text{QU}_M) \cdot \text{QU}_{M^*}$. This gives an isomorphism of the corresponding flows. \square

Let \hat{L} be the continuous signature consisting of the metric d , a unary predicate P^M and the symbols of the language L . We construct an \hat{L} -expansion of \mathbb{U} , say \mathbb{U}_M , so that the zero-set of P^M is a discrete first-order structure which is isomorphic to M with respect to zero-sets of the continuous counterparts of the relations of L .

By Lemma 3.6 the Urysohn space \mathbb{U} contains $\mathbb{Q}\mathbb{U}_M$ as a dense subset. Using this we define the continuous structure \mathbb{U}_M as follows:

- (i) $P^M(u) = d(u, P^M(\mathbb{Q}\mathbb{U}))$,
- (ii) for each relational L -symbol R on P^M and a tuple \bar{u} of appropriate length let $R(\bar{u}) = d(\bar{u}, R(\mathbb{Q}\mathbb{U}))$.

As a result we have a continuous structure where continuity moduli are just id and the zero-set of any relation coincides with its counterpart from $\mathbb{Q}\mathbb{U}_M$. In particular the structure M is realised on the zero-set of P^M .

We will assume that the embedding of $\mathbb{Q}\mathbb{U}_M$ into \mathbb{U} satisfies conditions (i) - (iii) of the beginning of Section 3.2. In particular any automorphism of the continuous structure $\mathbb{Q}\mathbb{U}_M$ extends to an automorphism of \mathbb{U}_M and $Iso(\mathbb{Q}\mathbb{U}_M)$ is dense in $Iso(\mathbb{U}_M)$.

The group $Iso(\mathbb{U}_M)$ has a natural actions on the space $Exp(M, Age(M^*))$ of all $Age(M^*)$ -expansions of M . These action is defined by restriction to the zero-set of P^M .

Proposition 3.9 *If $Iso(\mathbb{Q}\mathbb{U}_M)$ has metrizable universal minimal flow, then the $Iso(\mathbb{U}_M)$ -space $Exp(M, Age(M^*))$ is a universal minimal flow of $Iso(\mathbb{U}_M)$.*

Proof. The minimality of $Exp(M, Age(M^*))$ follows from the fact that it is already minimal for $Iso(\mathbb{Q}\mathbb{U}_M)$.

To see that $Exp(M, Age(M^*))$ is universal take any $Iso(\mathbb{U}_M)$ -flow C . Since C is an $Iso(\mathbb{Q}\mathbb{U}_M)$ -flow, by Lemma 3.8 there is an $Iso(\mathbb{Q}\mathbb{U}_M)$ -morphism from $Exp(M, Age(M^*))$ to C . Since $Iso(\mathbb{Q}\mathbb{U}_M)$ is dense in $Iso(\mathbb{U}_M)$ this morphism is $Iso(\mathbb{U}_M)$ -equivariant. \square

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